COMMUTATIVE ENERGETIC SUBSETS OF
$BCK$-ALGEBRAS

Abstract

The notions of a $C$-energetic subset and (anti) permeable $C$-value in $BCK$-algebras are introduced, and related properties are investigated. Conditions for an element $t$ in $[0, 1]$ to be an (anti) permeable $C$-value are provided. Also conditions for a subset to be a $C$-energetic subset are discussed. We decompose $BCK$-algebra by a partition which consists of a $C$-energetic subset and a commutative ideal.

Keywords: $S$-energetic subset, $I$-energetic subset, $C$-energetic subset, (anti) fuzzy commutative ideal, (anti) permeable $I$-value, (anti) permeable $C$-value.

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1. Introduction

Jun et al. [3] introduced the notions of energetic (resp. right vanished, right stable) subsets and (anti) permeable values in $BCK/BCI$-algebras. Using the notion of (anti) fuzzy subalgebras/ideals of $BCK/BCI$-algebras, they investigated relations among subalgebras/ideals, energetic subsets, (anti) permeable values, right vanished subsets and right stable subsets.

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In this article, we introduce the notions of a $C$-energetic subset and (anti) permeable $C$-value in $BCK$-algebras, and investigate related properties. We provide conditions for an element $t$ in $[0, 1]$ to be an (anti) permeable $C$-value. We also discuss conditions for a subset to be a $C$-energetic subset. We show that a $BCK$-algebra is decomposed by a partition which consists of a $C$-energetic subset and a commutative ideal.

2. Preliminaries

A $BCK/BCI$-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a $BCI$-algebra it satisfies the following conditions

(I) $(\forall x, y, z \in X) \ ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0$,
(II) $(\forall x, y \in X) \ ((x \ast (x \ast y)) \ast y = 0)$,
(III) $(\forall x \in X) \ (x \ast x = 0)$,
(IV) $(\forall x, y \in X) \ (x \ast y = 0, y \ast x = 0 \Rightarrow x = y)$.

If a $BCI$-algebra $X$ satisfies the following identity

(V) $(\forall x \in X) \ (0 \ast x = 0)$,

then $X$ is called a $BCK$-algebra. Any $BCK/BCI$-algebra $X$ satisfies the following axioms

$$
(\forall x \in X) \ (x \ast 0 = x), \tag{2.1}
$$
$$
(\forall x, y, z \in X) \ (x \leq y \Rightarrow x \ast z \leq y \ast z, z \ast y \leq z \ast x), \tag{2.2}
$$
$$
(\forall x, y, z \in X) \ ((x \ast y) \ast z = (x \ast z) \ast y), \tag{2.3}
$$
$$
(\forall x, y, z \in X) \ ((x \ast z) \ast (y \ast z) \leq x \ast y), \tag{2.4}
$$

where $x \leq y$ if and only if $x \ast y = 0$. A nonempty subset $S$ of a $BCK/BCI$-algebra $X$ is called a subalgebra of $X$ if $x \ast y \in S$ for all $x, y \in S$. A subset $I$ of a $BCK/BCI$-algebra $X$ is called an ideal of $X$ if it satisfies

$$
0 \in I, \tag{2.5}
$$
$$
(\forall x \in X) \ (\forall y \in I) \ (x \ast y \in I \Rightarrow x \in I). \tag{2.6}
$$

A subset $I$ of a $BCK$-algebra $X$ is called a commutative ideal (see [5]) of $X$ if it satisfies (2.5) and

$$
(\forall x, y \in X) \ (\forall z \in I) \ ((x \ast y) \ast z \in I \Rightarrow x \ast (y \ast (y \ast x)) \in I). \tag{2.7}
$$
Observe that every commutative ideal is an ideal, but the converse is not true (see [6]).

We refer the reader to the books [2, 6] for further information regarding BCK/BCI-algebras.

The concept of fuzzy sets was introduced by Zadeh [7]. Let \( X \) be a set. The mapping \( f : X \rightarrow [0, 1] \) is called a fuzzy set in \( X \).

A fuzzy set \( f \) in a BCK/BCI-algebra \( X \) is called a fuzzy subalgebra of \( X \) if it satisfies

\[
(\forall x, y \in X) \ (f(x * y) \geq \min\{f(x), f(y)\}). \tag{2.8}
\]

A fuzzy set \( f \) in a BCK/BCI-algebra \( X \) is called a fuzzy ideal of \( X \) if it satisfies

\[
(\forall x \in X) \ (f(0) \geq f(x)). \tag{2.9}
\]
\[
(\forall x, y \in X) \ (f(x) \geq \min\{f(x * y), f(y)\}). \tag{2.10}
\]

Note that every fuzzy ideal \( f \) of a BCK/BCI-algebra \( X \) satisfies

\[
(\forall x, y \in X) \ (x \leq y \Rightarrow f(x) \geq f(y)). \tag{2.11}
\]

A fuzzy set \( f \) in a BCK-algebra \( X \) is called a fuzzy commutative ideal (see [4]) of \( X \) if it satisfies (2.9) and

\[
(\forall x, y, z \in X) \ (f(x * (y * (y * x))) \geq \min\{f((x * y) * z), f(z)\}). \tag{2.12}
\]

For a fuzzy set \( f \) in \( X \) and \( t \in [0, 1] \), the (strong) upper (resp. lower) \( t \)-level sets are defined as follows:

\[
U(f; t) := \{x \in X \mid f(x) \geq t\}, \quad U^*(f; t) := \{x \in X \mid f(x) > t\},
\]
\[
L(f; t) := \{x \in X \mid f(x) \leq t\}, \quad L^*(f; t) := \{x \in X \mid f(x) < t\}.
\]

### 3. Commutative energetic subsets

In what follows, let \( X \) denote a BCK-algebra unless otherwise specified.

**Definition 3.1 ([3])**. A non-empty subset \( A \) of \( X \) is said to be S-energetic if it satisfies

\[
(\forall a, b \in X) \ (a * b \in A \Rightarrow \{a, b\} \cap A \neq \emptyset). \tag{3.1}
\]

**Definition 3.2 ([3])**. A non-empty subset \( A \) of \( X \) is said to be I-energetic if it satisfies

\[
(\forall x, y \in X) \ (y \in A \Rightarrow \{x, y * x\} \cap A \neq \emptyset). \tag{3.2}
\]
Lemma 3.3 ([3]). For any subset $A$ of $X$, if $X \setminus A$ is an ideal of $X$, then $A$ is $I$-energetic.

Definition 3.4. A non-empty subset $A$ of $X$ is said to be commutative energetic (briefly, $C$-energetic) if it satisfies

$$(\forall x, y, z \in X) (x \ast (y \ast (y \ast x)) \in A \Rightarrow \{z, (x \ast y) \ast z\} \cap A \neq \emptyset). \quad (3.3)$$

Example 3.5. Let $X = \{0, 1, 2, 3, 4\}$ be a $BCK$-algebra with the following Cayley table

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It is routine to verify that $A := \{3, 4\}$ is a $C$-energetic subset of $X$.

We consider relations between an $I$-energetic subset and a $C$-energetic subset.

Theorem 3.6. Every $C$-energetic subset is $I$-energetic.

Proof: Let $A$ be a $C$-energetic subset of $X$. Let $x, y \in X$ be such that $y \in A$. Then $y \ast (0 \ast (0 \ast y)) = y \in A$, and so $\{x, (y \ast 0) \ast x\} \cap A \neq \emptyset$ by (3.3). It follows from (2.1) that $\{x, y \ast x\} \cap A \neq \emptyset$. Hence $A$ is an $I$-energetic subset of $X$.

The converse of Theorem 3.6 is not true as seen in the following examples.

Example 3.7. Let $X = \{0, 1, 2, 3, 4\}$ be a $BCK$-algebra with the following Cayley table

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Take $A := \{1, 2, 4\}$. Then $X \setminus A = \{0, 3\}$ is an ideal of $X$. Hence, by Lemma 3.3, $A$ is an $I$-energetic subset of $X$. But it is not $C$-energetic since
Since strong lower Corollary

Theorem 3.8. For any nonempty subset $A$ of $X$, if $X \setminus A$ is a commutative ideal of $X$, then $A$ is $C$-energetic.

Proof: Assume that $A$ is not $C$-energetic. Then for any $x, y \in X$ with

$$x \ast (y \ast (y \ast x)) \in A,$$

there exists $z \in X$ such that \( \{z, (x \ast y) \ast z\} \cap A = \emptyset \). It follows that

$$(x \ast y) \ast z \in X \setminus A \quad \text{and} \quad z \in X \setminus A.$$  

Since $X \setminus A$ is a commutative ideal of $X$, we have $x \ast (y \ast (y \ast x)) \in X \setminus A$, that is, $x \ast (y \ast (y \ast x)) \notin A$. This is a contradiction, and so $A$ is a $C$-energetic subset of $X$. \( \square \)

Corollary 3.9. For any nonempty subset $A$ of $X$, if $X \setminus A$ is a commutative ideal of $X$, then $A$ is $I$-energetic.

Theorem 3.10. Let $A$ be a nonempty subset of $X$ with $0 \notin A$. If $A$ is $C$-energetic, then $X \setminus A$ is a commutative ideal of $X$.

Proof: Obviously $0 \in X \setminus A$. Let $x, y, z \in X$ be such that $z \in X \setminus A$ and $(x \ast y) \ast z \in X \setminus A$. Assume that $x \ast (y \ast (y \ast x)) \in A$. Then \( \{z, (x \ast y) \ast z\} \cap A \neq \emptyset \) by (3.3), which implies that $z \in A$ or $(x \ast y) \ast z \in A$. This is a contradiction, and so $x \ast (y \ast (y \ast x)) \in X \setminus A$. This shows that $X \setminus A$ is a commutative ideal of $X$. \( \square \)

Corollary 3.11. Let $A$ be a nonempty subset of $X$ with $0 \notin A$. If $A$ is $C$-energetic, then $X \setminus A$ is an ideal and hence a subalgebra of $X$.

Theorem 3.12. If $f$ is a fuzzy commutative ideal of $X$, then the nonempty lower $t$-level set $L(f; t)$ is a $C$-energetic subset of $X$.

Proof: Assume that $L(f; t) \neq \emptyset$ for $t \in [0, 1]$. Let $x, y \in X$ be such that $x \ast (y \ast (y \ast x)) \in L(f; t)$. Then

$$t \geq f(x \ast (y \ast (y \ast x))) \geq \min\{f((x \ast y) \ast z), f(z)\}$$

for all $z \in X$, which implies that $f((x \ast y) \ast z) \leq t$ or $f(z) \leq t$, that is, $(x \ast y) \ast z \in L(f; t)$ or $z \in L(f; t)$. Thus \( \{z, (x \ast y) \ast z\} \cap L(f; t) \neq \emptyset \), and therefore $L(f; t)$ is a $C$-energetic subset of $X$. \( \square \)

Corollary 3.13. If $f$ is a fuzzy commutative ideal of $X$, then the nonempty strong lower $t$-level set $L^*(f; t)$ is a $C$-energetic subset of $X$.

Since $L(f; t) \cup U^*(f; t) = X$ and $L(f; t) \cap U^*(f; t) = \emptyset$ for all $t \in [0, 1]$, we have the following corollary.
Corollary 3.14. If $f$ is a fuzzy commutative ideal of $X$, then $U^*(f; t)$ is empty or a commutative ideal of $X$ for all $t \in [0, 1]$.

Definition 3.15 ([1]). A fuzzy set $f$ in $X$ is called an anti fuzzy ideal of $X$ if it satisfies

$$\forall x \in X \ (f(0) \leq f(x)). \ \ (3.4)$$

$$\forall x, y \in X \ (f(x) \leq \max\{f(x * y), f(y)\}). \ \ (3.5)$$

Definition 3.16. A fuzzy set $f$ in $X$ is called an anti fuzzy commutative ideal of $X$ if it satisfies (3.4) and

$$\forall x, y, z \in X \ (f(x * (y * (y * x))) \leq \max\{f((x * y) * z), f(z)\}). \ \ (3.6)$$

Example 3.17. Consider a BCK-algebra $X = \{0, a, b, c\}$ with the following Cayley table

\[
\begin{array}{c|cccc}
  * & 0 & a & b & c \\
\hline
  0 & 0 & 0 & 0 & 0 \\
  a & a & 0 & 0 & a \\
b & b & a & 0 & b \\
c & c & c & c & 0 \\
\end{array}
\]

Define a fuzzy set $f$ in $X$ as follows

$$f : X \to [0, 1], \ x \mapsto \begin{cases} 
  t_0 & \text{if } x = 0, \\
  t_1 & \text{if } x = c, \\
  t_2 & \text{if } x \in \{a, b\}
\end{cases}$$

where $t_0 < t_1 < t_2$ in $[0, 1]$. It is routine to verify that $f$ is an anti fuzzy commutative ideal of $X$.

Theorem 3.18. Every anti fuzzy commutative ideal is an anti fuzzy ideal.

Proof: Let $f$ be an anti fuzzy commutative ideal of $X$. If we put $y = 0$ in (3.6), then

$$\max\{f(x * z), f(z)\} = \max\{f((x * 0) * z), f(z)\} \geq f(x * (0 * (0 * x))) = f(x).$$

Hence $f$ is an anti fuzzy ideal of $X$. \qed

The converse of Theorem 3.18 is not true as seen in the following example.
Example 3.19. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table

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Define a fuzzy set $f$ in $X$ as follows

$$f : X \rightarrow [0, 1], \quad x \mapsto \begin{cases} s_0 & \text{if } x = 0, \\ s_1 & \text{if } x = 1, \\ s_2 & \text{if } x \in \{2, 3, 4\} \end{cases}$$

where $s_0 < s_1 < s_2$ in $[0, 1]$. Then $f$ is an anti fuzzy ideal of $X$. But it is not an anti fuzzy commutative ideal of $X$ since

$$f(2 \ast (3 \ast (3 \ast 2))) \not\leq \max\{f(0), f((2 \ast 3) \ast 0)\}.$$  

We provide a characterization of an anti fuzzy commutative ideal.

Theorem 3.20. For a fuzzy set $f$ in $X$, the following are equivalent.

1. $f$ is an anti fuzzy commutative ideal of $X$.
2. $f$ is an anti fuzzy ideal of $X$ satisfying the following condition

$$(\forall x, y \in X) \ (f(x \ast (y \ast (y \ast x)))) \leq f(x \ast y)). \quad (3.7)$$

Proof: Assume that $f$ is an anti fuzzy commutative ideal of $X$. Then $f$ is an anti fuzzy ideal of $X$ (see Theorem 3.18). Taking $z = 0$ in (3.6) and using (3.4) and (2.1), we have (3.7).

Conversely, suppose that (2) is valid. Then

$$f(x \ast y) \leq \max\{f((x \ast y) \ast z), f(z)\} \quad (3.8)$$

for all $x, y, z \in X$. Combining (3.7) and (3.8), we get (3.6). The proof is complete.

Definition 3.21 ([3]). Let $f$ be a fuzzy set in $X$. A number $t \in [0, 1]$ is called a permeable I-value for $f$ if $U(f; t) \neq \emptyset$ and the following assertion is valid.

$$(\forall x, y \in X) \ (f(y) \geq t \Rightarrow \max\{f(y \ast x), f(x)\} \geq t). \quad (3.9)$$
**Definition 3.22.** Let $f$ be a fuzzy set in $X$. A number $t \in [0,1]$ is called a permeable $C$-value for $f$ if $U(f; t) \neq \emptyset$ and the following assertion is valid.

$$f(x \ast (y \ast (y \ast x))) \geq t \Rightarrow \max\{f((x \ast y) \ast z), f(z)\} \geq t \quad (3.10)$$

for all $x, y, z \in X$.

**Example 3.23.** Consider a BCK-algebra $X = \{0, a, b, c\}$ which is given in Example 3.17. Let $f$ be a fuzzy set in $X$ defined by $f(0) = 0.3$, $f(a) = f(b) = 0.7$ and $f(c) = 0.5$. If $t \in (0.5, 0.7]$, then $U(f; t) = \{a, b\}$ and it is easy to check that $t$ is a permeable $C$-value for $f$.

**Theorem 3.24.** Let $f$ be a fuzzy commutative ideal of $X$. If $t \in [0,1]$ is a permeable $C$-value for $f$, then the nonempty upper $t$-level set $U(f; t)$ is a $C$-energetic subset of $X$.

**Proof:** Assume that $U(f; t) \neq \emptyset$ for $t \in [0,1]$. Let $x, y \in X$ be such that $x \ast (y \ast (y \ast x)) \in U(f; t)$. Then $f(x \ast (y \ast (y \ast x))) \geq t$, and so $\max\{f((x \ast y) \ast z), f(z)\} \geq t$ by (3.10). It follows that $f((x \ast y) \ast z) \geq t$ or $f(z) \geq t$, that is, $(x \ast y) \ast z \in U(f; t)$ or $z \in U(f; t)$. Hence $\{z, (x \ast y) \ast z\} \cap U(f; t) \neq \emptyset$, and therefore $U(f; t)$ is a $C$-energetic subset of $X$.

Since $U(f; t) \cup L^*(f; t) = X$ and $U(f; t) \cap L^*(f; t) = \emptyset$ for all $t \in [0,1]$, we have the following corollary.

**Corollary 3.25.** Let $f$ be a fuzzy commutative ideal of $X$. If $t \in [0,1]$ is a permeable $C$-value for $f$, then $L^*(f; t)$ is empty or a commutative ideal of $X$.

**Theorem 3.26.** For a fuzzy set $f$ in $X$, if there exists a subset $K$ of $[0,1]$ such that $\{U(f; t), L^*(f; t)\}$ is a partition of $X$ and $L^*(f; t)$ is a commutative ideal of $X$ for all $t \in K$, then $t$ is a permeable $C$-value for $f$.

**Proof:** Assume that $f(x \ast (y \ast (y \ast x))) \geq t$ for any $x, y \in X$. Then $x \ast (y \ast (y \ast x)) \in U(f; t)$, and so $\{z, (x \ast y) \ast z\} \cap U(f; t) \neq \emptyset$ since $U(f; t)$ is a $C$-energetic subset of $X$. It follows that $z \in U(f; t)$ or $(x \ast y) \ast z \in U(f; t)$ and so that

$$\max\{f((x \ast y) \ast z), f(z)\} \geq t.$$ 

Therefore $t$ is a permeable $C$-value for $f$.

**Theorem 3.27.** Let $f$ be a fuzzy set in $X$ with $U(f; t) \neq \emptyset$ for $t \in [0,1]$. If $f$ is an anti fuzzy commutative ideal of $X$, then $t$ is a permeable $C$-value for $f$. 

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Proof: Let \( x, y, z \in X \) be such that \( f(x \ast (y \ast (y \ast x))) \geq t \). Then
\[
t \leq f(x \ast (y \ast (y \ast x))) \leq \max\{f((x \ast y) \ast z), f(z)\}
\]
by (3.6). Hence \( t \) is a permeable \( C \)-value for \( f \).

Theorem 3.28. If \( f \) is an anti fuzzy commutative ideal of \( X \), then
\[
(\forall t \in [0, 1]) \ (U(f; t) \neq \emptyset \Rightarrow U(f; t) \text{ is a } C\text{-energetic subset of } X).
\]
Proof: Let \( x, y, z \in X \) be such that \( x \ast (y \ast (y \ast x)) \in U(f; t) \). Then
\[
f(x \ast (y \ast (y \ast x))) \geq t,
\]
which implies from (3.6) that
\[
t \leq f(x \ast (y \ast (y \ast x))) \leq \max\{f((x \ast y) \ast z), f(z)\}.
\]
Hence \( f((x \ast y) \ast z) \geq t \) or \( f(z) \geq t \), that is, \( (x \ast y) \ast z \in U(f; t) \) or \( z \in U(f; t) \). Thus \( \{z, (x \ast y) \ast z\} \cap U(f; t) \neq \emptyset \), and therefore \( U(f; t) \) is a \( C \)-energetic subset of \( X \).

Theorem 3.29. For any fuzzy set \( f \) in \( X \), every permeable \( C \)-value for \( f \) is a permeable \( I \)-value for \( f \).
Proof: Let \( t \in [0, 1] \) be a permeable \( C \)-value for \( f \). Assume that \( f(y) \geq t \) for all \( y \in X \). Then
\[
t \leq f(y) = f((y \ast (0 \ast (0 \ast y)))
\]
by (V) and (2.1), and so
\[
t \leq \max\{f((y \ast 0) \ast z), f(z)\} = \max\{f(y \ast z), f(z)\}
\]
for all \( y, z \in X \) by (3.10) and (2.1). Therefore \( t \) is a permeable \( I \)-value for \( f \).

Definition 3.30 ([3]). Let \( f \) be a fuzzy set in \( X \). A number \( t \in [0, 1] \) is called an anti permeable \( I \)-value for \( f \) if \( L(f; t) \neq \emptyset \) and the following assertion is valid.
\[
(\forall x, y \in X) \ (f(y) \leq t \Rightarrow \min\{f(y \ast x), f(x)\} \leq t). \tag{3.11}
\]

Theorem 3.31. Let \( f \) be a fuzzy set in \( X \) with \( L(f; t) \neq \emptyset \) for \( t \in [0, 1] \). If \( f \) is a fuzzy ideal of \( X \), then \( t \) is an anti permeable \( I \)-value for \( f \).
Proof: Let \( f(y) \leq t \) for \( y \in X \). Then
\[
\min\{f(y \ast x), f(x)\} \leq f(y) \leq t
\]
for all $x \in X$ by (2.10). Hence $t$ is an anti permeable $I$-value for $f$. □

**Definition 3.32.** Let $f$ be a fuzzy set in $X$. A number $t \in [0, 1]$ is called an anti permeable $C$-value for $f$ if $L(f; t) \neq \emptyset$ and the following assertion is valid.

$$f(x \ast (y \ast (y \ast x))) \leq t \Rightarrow \min \{f((x \ast y) \ast z), f(z)\} \leq t \quad (3.12)$$

for all $x, y, z \in X$.

**Theorem 3.33.** Let $f$ be a fuzzy set in $X$ with $L(f; t) \neq \emptyset$ for $t \in [0, 1]$. If $f$ is a fuzzy commutative ideal of $X$, then $t$ is an anti permeable $C$-value for $f$.

**Proof:** Let $x, y \in X$ be such that $f(x \ast (y \ast (y \ast x))) \leq t$. Then

$$\min \{f((x \ast y) \ast z), f(z)\} \leq f(x \ast (y \ast (y \ast x))) \leq t$$

for all $z \in X$ by (2.12). Hence $t$ is an anti permeable $C$-value for $f$. □

**Theorem 3.34.** Let $f$ be an anti fuzzy commutative ideal of $X$. If $t \in [0, 1]$ is an anti permeable $C$-value for $f$, then the lower $t$-level set $L(f; t)$ is a $C$-energetic subset of $X$.

**Proof:** Let $x, y \in X$ be such that $x \ast (y \ast (y \ast x)) \in L(f; t)$. Then $f(x \ast (y \ast (y \ast x))) \leq t$ and so $\min \{f((x \ast y) \ast z), f(z)\} \leq t$ by (3.12). It follows that $(x \ast y) \ast z \in L(f; t)$ or $z \in L(f; t)$. Hence $\{z, (x \ast y) \ast z\} \cap L(f; t) \neq \emptyset$, and therefore $L(f; t)$ is a $C$-energetic subset of $X$. □

**Corollary 3.35.** Let $f$ be an anti fuzzy commutative ideal of $X$. If $t \in [0, 1]$ is an anti permeable $C$-value for $f$, then $U^*(f; t)$ is empty or a commutative ideal of $X$.

**Theorem 3.36.** For a fuzzy set $f$ in $X$, if there exists a subset $K$ of $[0, 1]$ such that $\{U^*(f; t), L(f; t)\}$ is a partition of $X$ and $U^*(f; t)$ is a commutative ideal of $X$ for all $t \in K$, then $t$ is an anti permeable $C$-value for $f$.

**Proof:** Assume that $f(x \ast (y \ast (y \ast x))) \leq t$ for any $x, y \in X$. Then

$$x \ast (y \ast (y \ast x)) \in L(f; t),$$

and so $\{z, (x \ast y) \ast z\} \cap L(f; t) \neq \emptyset$ for all $z \in X$ since $L(f; t)$ is a $C$-energetic subset of $X$. It follows that $f(z) \leq t$ or $f((x \ast y) \ast z) \leq t$, and so that

$$\min \{f((x \ast y) \ast z), f(z)\} \leq t.$$

Therefore $t$ is an anti permeable $C$-value for $f$. □
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